

# ON A CLASS OF CONSERVATION RULES ASSOCIATED WITH SENSITIVITY ANALYSIS IN LINEAR ELASTICITY

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**Abstract**—Considering arbitrary stress, strain or displacement functionals specified over a domain of an elastic, homogeneous and isotropic body, their invariance is proved for the case of translation, rotation and scale change of an arbitrary domain within the body. The associated class of path-independent integrals is derived. It is shown that sensitivity analysis with respect to translation, rotation or expansion of defects can be performed by using these path-independent integrals.

## 1. INTRODUCTION

The present paper discusses a new class of conservation rules which constitute an extension of the class considered by Eshelby[1, 2], Günther[3], Knowles and Sternberg[4], Gołębiewska-Herrmann[7], Delph[9], Rice[12] and Bui[14] for linear and nonlinear elasticity. Whereas the previous rules were associated with the potential or complementary energy variation due to translation, rotation or scale change of the body, the present analysis is concerned with an arbitrary functional of stress, strain or displacement. Similar to the Eshelby[1, 2] or Budiansky and Rice[13] interpretation, the variation of respective functionals can be interpreted as that corresponding to translation, rotation or size variation of inhomogeneities within the body. Therefore the derived conservation rules can find application in identification problems where the location and size of defects are to be determined for some mechanical measurements, or in studying the variation of global or local body response due to variation of position and size of defects such as cracks, voids and inclusions. The general formulation of such a sensitivity analysis problem was presented in previous papers by Dems and Mróz[15, 16], and here this analysis will be extended by discussing three types of path-independent integrals and their interpretations.

In discussing the conservation rules, Knowles and Sternberg[4] demonstrated that such laws follow from Noether's theorem[6] on invariant variational principles combined with the principle of stationary potential energy. In our case, the functionals considered do not possess this stationarity property and therefore the derived conservation rules are not directly generated from Noether's theorem,† thus constituting a new class of rules. The concept of primary and adjoint systems will be used and the conservation rules will be expressed in terms of stress and strain fields of both systems. A somewhat similar idea of introducing adjoint variables was discussed recently by Herrmann[11] and Delph[10] who considered a nonlinear creep problem for which an energy stationarity principle does not exist. In particular, it will be shown that bilinear functionals of primary and adjoint variables can also be considered within the considered class of functionals. Since the local displacement or stress components can be expressed as bilinear functionals, their variation can be derived through the use of path-independent integrals. The conservation rules for the mutual potential energy of two equilibrium states were discussed by Chen and Shield[18] and applied in fracture mechanics for the determination of stress intensity factors  $K^I$ ,  $K^{II}$

† They can be derived from Noether's theorem by considering augmented functionals which take into account the equilibrium and compatibility equations of the body.

in two modes. The same problem was reanalysed by Bui[19], who proposed an alternative method of solution using the associated path-independent integrals.

The conservation rules in elastodynamics were discussed by Fletcher[5], whereas Francfort and Gołębiewska-Herrmann[8] derived the conservation laws in thermoelasticity using the convolution products of primary and adjoint states. It can be shown that the present approach can easily be extended to elastodynamics and to time-dependent problems, but in the present paper we limit our analysis to the case of elastostatics.

In Section 2, the general expression for variation of arbitrary volume or surface integrals of stress, strain and displacements due to boundary variation will be derived, and the concept of an adjoint body will be introduced following the previous works by Dems and Mróz[15, 16]. In Section 3, the respective conservation rules will be proved for the case of linear elasticity and small strain theory. In Section 4, the transition to conservation laws discussed in Refs. [1-4, 14, 18] will be performed, whereas in Section 6 some possible applications will be indicated.

## 2. VARIATION OF VOLUME AND SURFACE INTEGRALS DUE TO BOUNDARY VARIATION

Consider a linear elastic body with surface tractions  $\mathbf{T}^0$  specified on its boundary portion  $S_T$  and displacements  $\mathbf{u}^0$  prescribed on the portion  $S_u$ , where  $S = S_T \cup S_u$  denotes the boundary of the body. We shall discuss the variation of functionals

$$G_1 = \int \psi(\boldsymbol{\sigma}, \mathbf{u}) \, dV + \int h(\mathbf{T}, \mathbf{u}) \, dS \quad (1)$$

or

$$G_2 = \int \phi(\boldsymbol{\varepsilon}, \mathbf{u}) \, dV + \int h(\mathbf{T}, \mathbf{u}) \, dS \quad (2)$$

specified over the body domain of volume  $V$ , associated with the variation of its boundaries. Here  $\psi(\boldsymbol{\sigma}, \mathbf{u})$ ,  $\phi(\boldsymbol{\varepsilon}, \mathbf{u})$  and  $h(\mathbf{T}, \mathbf{u})$  are continuous and differentiable functions of their arguments. The stress, strain and displacement fields are denoted by  $\boldsymbol{\sigma}$ ,  $\boldsymbol{\varepsilon}$  and  $\mathbf{u}$ , where  $\mathbf{u}(\mathbf{x})$  is a continuous field and  $\boldsymbol{\sigma}(\mathbf{x})$ ,  $\boldsymbol{\varepsilon}(\mathbf{x})$  are piecewise continuous fields. The particular case when  $\psi(\boldsymbol{\sigma}, \mathbf{u}) = \psi_1(\boldsymbol{\sigma}) + \psi_2(\mathbf{u})$ ,  $h(\mathbf{T}, \mathbf{u}) = h_1(\mathbf{T}) + h_2(\mathbf{u})$ ,  $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = \phi_1(\boldsymbol{\varepsilon}) + \phi_2(\mathbf{u})$  was discussed in previous papers[15,16]. We assume this particular case in Section 3 when considering rotation and scale change of body domain.

The variation of body shape is conceived as the transformation process specified by the transformation field  $\boldsymbol{\varphi}(\mathbf{x})$  mapping the material points from an initial to a transformed configuration,  $P \rightarrow P^*$ :  $\mathbf{x}_i = \mathbf{x} + \boldsymbol{\varphi}$ . In this paper, we shall restrict our analysis to an infinitesimal transformation  $\delta\boldsymbol{\varphi}(\mathbf{x})$  from the assumed configuration and derive the formulae for variation of the functionals  $G_1$  and  $G_2$  associated with this transformation. If  $\mathbf{x}^*$  denotes the position of a point  $P$ , initially placed at  $\mathbf{x}$ , after infinitesimal variation  $\delta\boldsymbol{\varphi}(\mathbf{x})$ , we have

$$P \rightarrow P^*: x_i^* = x_i + \delta\varphi_i \quad (3)$$

and the variations of displacement, stress and strain fields are expressed in a fixed reference system as follows:

$$\begin{aligned} \delta u_i &= u_i^*(\mathbf{x}^*) - u_i(\mathbf{x}) = \delta \bar{u}_i + u_{i,k} \delta \varphi_k, \\ \delta \varepsilon_{ij} &= \delta \bar{\varepsilon}_{ij} + \varepsilon_{ij,k} \delta \varphi_k, \quad \delta \dot{\sigma}_{ij} = \delta \bar{\sigma}_{ij} + \sigma_{ij,k} \delta \varphi_k, \end{aligned} \quad (4)$$

where  $\delta \bar{\mathbf{u}}$ ,  $\delta \bar{\boldsymbol{\varepsilon}}$  and  $\delta \bar{\boldsymbol{\sigma}}$  denote the variations at the initial configuration of the body, and subscripts following commas denote partial derivatives with respect to coordinates of the Cartesian system. Clearly, the variations  $\delta \bar{\mathbf{u}}$ ,  $\delta \bar{\boldsymbol{\varepsilon}}$ ,  $\delta \bar{\boldsymbol{\sigma}}$  can be determined by considering an

incremental boundary value problem which accounts for the variation of boundary conditions due to the boundary surface transformation. The variations of surface tractions and of volume and surface elements are, cf. Ref. [16],

$$\begin{aligned} \delta T_i(\mathbf{x}) &= \delta \bar{T}_i(\mathbf{x}) + (\sigma_{ij}n_jn_l - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k, \\ \delta[dV(\mathbf{x})] &= \delta\varphi_{k,k} dV, \\ \delta[dS(\mathbf{x})] &= (\delta_{kl} - n_kn_l)\delta\varphi_{k,l}, \end{aligned} \tag{5}$$

where  $\mathbf{n}$  denotes the unit normal vector to the boundary surface.

In view of (5), the variation of the functional  $G_1$  corresponding to an infinitesimal transformation of the body domain is expressed as follows :

$$\begin{aligned} \delta G_1 &= \int \left( \frac{\partial\psi}{\partial\sigma} \cdot \delta\bar{\sigma} + \frac{\partial\psi}{\partial\mathbf{u}} \cdot \delta\bar{\mathbf{u}} \right) dV + \int \psi n_k \delta\varphi_k dS + \int \left\{ \frac{\partial h}{\partial\mathbf{u}} \cdot \delta\bar{\mathbf{u}} + \frac{\partial h}{\partial\mathbf{T}} \cdot \mathbf{u}_{,k} \delta\varphi_k \right. \\ &\quad + \frac{\partial h}{\partial\mathbf{T}} \cdot \delta\bar{\mathbf{T}} + \frac{\partial h}{\partial T_i} [(\sigma_{ij}n_jn_k - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k] \\ &\quad \left. + h(\delta_{kl} - n_kn_l)\delta\varphi_{k,l} \right\} dS. \end{aligned} \tag{6}$$

The expression for  $\delta G_1$  contains the variations  $\delta\bar{\mathbf{u}}(\mathbf{x})$ ,  $\delta\bar{\boldsymbol{\sigma}}(\mathbf{x})$  and  $\delta\bar{\mathbf{T}}(\mathbf{x})$ , which should be determined from an additional boundary-value problem within the unperturbed domain which accounts for modification of the boundary conditions on  $S$ . This modification depends on a form of boundary variation. In fact, in view of (5), on  $S_T$  and  $S_u$  we have

$$\delta T_i^0 = \delta \bar{T}_i^0 + (\sigma_{ij}n_jn_l - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k, \quad \delta u_i^0 = \delta \bar{u}_i^0 + u_{i,k}\delta\varphi_k. \tag{7}$$

Assuming, for instance, configuration-independent loading and support conditions, we have  $\delta T_i^0 = \delta u_i^0 = 0$  and eqns (7) provide the values of  $\delta \bar{T}_i^0$  and  $\delta \bar{u}_i^0$  on  $S_T$  and  $S_u$  when boundary modification occurs, thus

$$\delta \bar{T}_i^0 = -(\sigma_{ij}n_jn_l - \sigma_{il})n_k\delta\varphi_{k,l} + \sigma_{ij,k}n_j\delta\varphi_k, \quad \delta \bar{u}_i^0 = -u_{i,k}\delta\varphi_k, \tag{8}$$

and (8) provide new boundary conditions for a problem of determining  $\delta\bar{\mathbf{u}}(\mathbf{x})$ ,  $\delta\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ ,  $\delta\bar{\boldsymbol{\sigma}}(\mathbf{x})$  within the body.

Such a direct approach may become impractical in cases where the form of boundary modification changes and numerous solutions are required in order to determine variations  $\delta\bar{\boldsymbol{\sigma}}$  and  $\delta\bar{\mathbf{u}}$ . An alternative approach used in sensitivity analysis requires introduction of an adjoint body and one solution of a boundary-value problem for this body. Following Refs. [15, 16], consider the adjoint elastic body of the same shape and material stress-strain relations, but satisfying the boundary conditions

$$\mathbf{T}^a = \frac{\partial h}{\partial\mathbf{u}} \quad \text{on } S_T, \quad \mathbf{u}^a = -\frac{\partial h}{\partial\mathbf{T}} \quad \text{on } S_u, \tag{9}$$

and with imposed body force and initial strain fields

$$\mathbf{f}^a = \frac{\partial\psi}{\partial\mathbf{u}}, \quad \boldsymbol{\varepsilon}^i = \frac{\partial\psi}{\partial\boldsymbol{\sigma}} \quad \text{within } V. \tag{10}$$

Denoting the stress within the adjoint body by  $\boldsymbol{\sigma}^a$ , its total strain field  $\boldsymbol{\varepsilon}^a$  can be presented as a sum, cf. Fig. 1

$$\boldsymbol{\varepsilon}^a = \boldsymbol{\varepsilon}^i + \boldsymbol{\varepsilon}^r \tag{11}$$

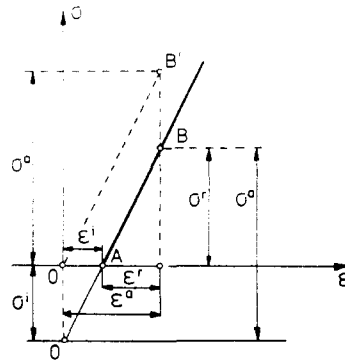


Fig. 1. Decomposition of strains and stresses in the adjoint body.

and is compatible with the displacement field  $\mathbf{u}^a$ . The stress field  $\boldsymbol{\sigma}^r$  is related to  $\boldsymbol{\varepsilon}^r$  by Hooke's law,  $\boldsymbol{\sigma}^r = \mathbf{D} \cdot \boldsymbol{\varepsilon}^r = \mathbf{D} (\boldsymbol{\varepsilon}^a - \boldsymbol{\varepsilon}^i)$ , and satisfies both equilibrium and boundary conditions

$$\operatorname{div} \boldsymbol{\sigma}^r + \mathbf{f}^a = 0 \quad \text{within } V, \quad \boldsymbol{\sigma}^r \cdot \mathbf{n} = \mathbf{T}^a \quad \text{on } S_T, \quad (12)$$

whereas the displacement field  $\mathbf{u}^a$  satisfies the boundary conditions  $\mathbf{u}^a = \mathbf{u}^a$  on  $S_u$ . In view of (9)–(12), the first two terms of (6) can be transformed as follows:

$$\begin{aligned} \int \left( \frac{\partial \psi}{\partial \boldsymbol{\sigma}} \cdot \delta \boldsymbol{\sigma} + \frac{\partial \psi}{\partial \mathbf{u}} \cdot \delta \bar{\mathbf{u}} \right) dV &= \int (\boldsymbol{\varepsilon}^i \cdot \delta \boldsymbol{\sigma} + \mathbf{f}^a \cdot \delta \bar{\mathbf{u}}) dV \\ &= \int (\boldsymbol{\varepsilon}^a \cdot \delta \boldsymbol{\sigma} - \boldsymbol{\varepsilon}^r \cdot \delta \boldsymbol{\sigma} + \mathbf{f}^a \cdot \delta \bar{\mathbf{u}}) dV \\ &= \int [\boldsymbol{\varepsilon}^a \cdot \delta \boldsymbol{\sigma} - (\boldsymbol{\sigma}^r \cdot \delta \boldsymbol{\varepsilon} - \mathbf{f}^a \cdot \delta \bar{\mathbf{u}})] dV \\ &= \int \mathbf{u}^a \cdot \delta \mathbf{T} dS - \int \mathbf{T}^a \cdot \delta \bar{\mathbf{u}} dS, \end{aligned} \quad (13)$$

where the following Betti's relations are applied:  $\boldsymbol{\varepsilon}^r \cdot \delta \boldsymbol{\sigma} = \boldsymbol{\varepsilon}^r \cdot \mathbf{D} \cdot \delta \boldsymbol{\varepsilon} = \boldsymbol{\sigma}^r \cdot \delta \boldsymbol{\varepsilon}$ , and  $\mathbf{D}$  is the elastic stiffness matrix of both the primary and adjoint bodies. Using (13), the expression (6) for  $\delta G_1$  can be presented in the form

$$\begin{aligned} \delta G_1 &= \int \left\{ \left[ \psi n_k + \frac{\partial h}{\partial u_i} u_{i,k} + \frac{\partial h}{\partial T_i} \sigma_{ij,k} n_j \right] \delta \varphi_k + \left[ h(\delta_{kl} - n_k n_l) \right. \right. \\ &\quad \left. \left. + \frac{\partial h}{\partial T_i} (\sigma_{ij} n_j n_l - \sigma_{il}) n_k \right] \delta \varphi_{k,l} \right\} dS + \int \left( \frac{\partial h}{\partial u_i} - \sigma_{ij} n_j \right) \delta \bar{u}_i^0 dS_u \\ &\quad + \int \left( \frac{\partial h}{\partial T_i} + u_i^a \right) \delta \bar{T}_i^0 dS_T, \end{aligned} \quad (14)$$

where the local variations  $\delta \bar{u}_i^0$  on  $S_u$  and  $\delta \bar{T}_i^0$  on  $S_T$  are specified by (7). Let us note that  $\delta \bar{u}_i^0$  and  $\delta \bar{T}_i^0$  are known on the boundary portions  $S_u$  and  $S_T$ . The expression (14) for  $\delta G_1$  now depends on stress and displacement fields of both the primary and adjoint bodies.

The variation of the functional  $G_2$  specified by (2) is expressed similarly as in the previous case. Introducing the adjoint body subjected to the boundary conditions specified

by (9) and with the initial stress and body force fields  $\sigma^i, f^a$  within  $V$ , such that

$$\sigma^i = \mathbf{D} \cdot \varepsilon^i = \frac{\partial \phi}{\partial \varepsilon}, \quad f^a = \frac{\partial \phi}{\partial \mathbf{u}}, \tag{15}$$

we obtain

$$\begin{aligned} \delta G_2 = & \int \left\{ \left[ \phi n_k + \frac{\partial h}{\partial u_i} u_{i,k} + \frac{\partial h}{\partial T_i} \sigma_{ij,k} n_j \right] \delta \varphi_k + \left[ h(\delta_{kl} - n_k n_l) \right. \right. \\ & \left. \left. + \frac{\partial h}{\partial T_i} (\sigma_{ij} n_j n_l - \sigma_{il}) n_k \right] \delta \varphi_{k,l} \right\} dS + \int \left( \frac{\partial h}{\partial u_i} - \sigma'_{ij} n_j \right) \delta \bar{u}_i^0 dS_u \\ & + \int \left( \frac{\partial h}{\partial T_i} + u_i^0 \right) \delta \bar{T}_i^0 dS_T. \end{aligned} \tag{16}$$

The expressions (14) and (16) now constitute the foundation for our subsequent analysis, in which the variations of  $G_1$  and  $G_2$  associated with translation, rotation and scale change of the body domain will be considered.

### 3. VARIATION OF $G_1$ AND $G_2$ ASSOCIATED WITH TRANSLATION, ROTATION AND SCALE CHANGE OF BODY DOMAIN

#### 3.1. Translation of body domain

Consider the translation of body domain by the infinitesimal vector  $\delta \mathbf{a}$ , so that

$$\mathbf{x}^* = \mathbf{x} + \delta \mathbf{a}. \tag{17}$$

The surface tractions  $\mathbf{T}^0$  and boundary displacements  $\mathbf{u}^0$  are also translated correspondingly, thus

$$\begin{aligned} \delta \mathbf{T}^0 &= \delta \sigma \cdot \mathbf{n} + \sigma \cdot \delta \mathbf{n} = (\delta \bar{\sigma} + \sigma_{,k} \delta a_k) \cdot \mathbf{n} = 0 \quad \text{on } S_T, \\ \delta \mathbf{u}^0 &= \delta \bar{\mathbf{u}}^0 + \mathbf{u}_{,k} \delta a_k = 0 \quad \text{on } S_u. \end{aligned} \tag{18}$$

The local variations  $\delta \bar{T}_i^0$  and  $\delta \bar{u}_i^0$  are therefore expressed as follows:

$$\delta \bar{T}_i^0 = -\sigma_{ij,k} n_j \delta a_k \quad \text{on } S_T, \quad \delta \bar{u}_i^0 = -u_{i,k} \delta a_k \quad \text{on } S_u. \tag{19}$$

The expression (14) for  $\delta G_1$  now takes the form

$$\begin{aligned} \delta G_1 = & \left\{ \int \left[ \psi n_k \frac{\partial h}{\partial u_i} u_{i,k} + \frac{\partial h}{\partial T_i} \sigma_{ij,k} n_j \right] dS - \int \left( \frac{\partial h}{\partial u_i} - \sigma'_{ij} n_j \right) u_{i,k} dS_u \right. \\ & \left. - \int \left( \frac{\partial h}{\partial T_i} + u_i^0 \right) \sigma_{ij,k} n_j dS_T \right\} \delta a_k = \left\{ \int [\psi \delta_{kj} + \sigma'_{ij} u_{i,k} - \sigma_{ij,k} u_i^0] n_j dS \right\} \delta a_k \\ & + \left\{ \int \left( \frac{\partial h}{\partial u_i} - \sigma'_{ij} n_j \right) u_{i,k} dS_T + \int \left( \frac{\partial h}{\partial T_i} + u_i^0 \right) \sigma_{ij,k} n_j dS_u \right\} \delta a_k. \end{aligned} \tag{20}$$

The last two integrals of (20) vanish in view of (9), and finally the variation  $\delta G_1$  is expressed in the form

$$\delta G_1 = (Z^k_{i\tau})_S \delta a_k, \quad (k = 1, 2, 3), \tag{21}$$

where

$$(Z^k_{\Gamma})_S = \int [\psi \delta_{kj} + \sigma'_{ij} u_{i,k} - \sigma_{ij,k} u_i^a] n_j \, dS. \tag{22}$$

Using the equality

$$\int \sigma_{ij} u_{i,j}^a n_k \, dS = \int \sigma_{ij,k} u_i^a n_j \, dS + \int \sigma_{ij} u_{i,j}^a n_k \, dS, \tag{23}$$

the expression for  $Z^k_{\Gamma}$  can be presented in the equivalent form

$$(Z^k_{\Gamma})_S = \int [\psi \delta_{kj} - \sigma_{il} \varepsilon_{il}^a \delta_{kj} + \sigma_{ij} u_{i,k}^a + \sigma'_{ij} u_{i,k}] n_j \, dS, \tag{24}$$

containing only gradients of the displacement fields  $u_i$  and  $u_i^a$ .

**THEOREM 1:** For a linear elastic and *homogeneous* body, the integral  $(Z^k_{\Gamma})_S$  vanishes for any closed surface within the body, thus

$$(Z^k_{\Gamma})_S = \int [\psi \delta_{kj} - \sigma_{ij,k} u_i^a + \sigma'_{ij} u_{i,k}] n_j \, dS = 0, \quad (k = 1, 2, 3). \tag{25}$$

To prove this theorem, let us transform (22) into a volume integral and use (9)–(12), obtaining

$$\begin{aligned} \int [\psi \delta_{kj} - \sigma_{ij,k} u_i^a + \sigma'_{ij} u_{i,k}] n_j \, dS &= \int [\psi_{,k} - (\sigma_{ij,k} u_i^a)_j \\ &+ (\sigma'_{ij} u_{i,k})_j] \, dV = \int \left[ \frac{\partial \psi}{\partial \sigma_{ij}} \sigma_{ij,k} + \frac{\partial \psi}{\partial u_i} u_{i,k} - \sigma_{ij,k} u_i^a - \sigma_{ij,k} u_{i,j}^a \right. \\ &+ \sigma'_{ij,j} u_{i,k} + \sigma'_{ij} u_{i,j,k} \left. \right] \, dV = \int [\varepsilon_{ij}^a \sigma_{ij,k} + f_i^a u_{i,k} - \sigma_{ij,k} \varepsilon_{ij}^a \\ &- f_i^a u_{i,k} + \sigma'_{ij} \varepsilon_{ij,k}] \, dV = 0, \end{aligned} \tag{26}$$

since for a homogeneous body

$$\begin{aligned} \sigma'_{ij} \varepsilon_{ij,k} &= \sigma'_{ij} (C_{ijpq} \sigma_{pq})_{,k} = \sigma'_{ij} C_{ijpq} \sigma_{pq,k} = \varepsilon'_{pq} \sigma_{pq,k}, \\ (C_{ijpq})_{,k} &= (D_{ijpq})_{,k} = 0. \end{aligned} \tag{27}$$

Here  $C$  denotes the elastic compliance matrix of both the primary and adjoint bodies.

For a non-homogeneous body, the integral (22) according to (21) represents the variation of the functional  $G_1$  due to infinitesimal translation of the boundary with respect to inhomogeneity. Alternatively, we can consider the translation of inhomogeneity or internal void with the exterior boundary fixed, cf. Fig. 2. In Fig. 2(a), the exterior boundary does not vary and the void of surface  $S_0$  translates through the distance  $\delta a$  within the homogeneous material. The variation of  $G_1$  can now be calculated from (21) by considering the integral (22) or (24) along the void surface  $S_0$ . For the free surface  $S_0$  the expression (24) is simplified, namely

$$(Z^k_{\Gamma})_{S_0} = \int (\psi - \sigma_{ij} \varepsilon_{ij}^a) n_k \, dS_0. \tag{28}$$

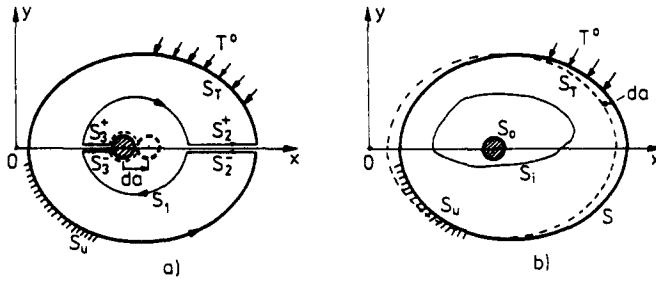


Fig. 2. (a) Translation of inhomogeneity within the body. (b) Translation of body with respect to fixed inhomogeneity.

Consider now the arbitrary closed surface  $S_1$  enclosing the cavity and connect it to the cavity surface  $S_0$  by the cuts  $S_2^+$  and  $S_2^-$ . Since the integral  $Z_{1T}^k$  taken along the surface  $S_1 + S_2^+ + S_0 + S_2^-$  vanishes and the integrals along  $S_2^+$  and  $S_2^-$  cancel, we obtain

$$(Z_{1T}^k)_{S_0} = (Z_{1T}^k)_{S_1} = (Z_{1T}^k)_S. \tag{29}$$

The transition from  $S_1$  to  $S$  can be performed by cuts  $S_2^+$  and  $S_2^-$ .

An alternative way to calculate the variation of  $G_1$  is to consider the translation of the body domain through the vector  $\delta a$  with the cavity fixed in space, cf. Fig. 2(b). The transition from the boundary surface  $S$  to an arbitrary closed surface  $S_1$  enclosing the cavity or to the cavity surface  $S_0$  is obtained by considering the cuts between these surfaces.

The variation of the functional  $G_2$  is expressed similarly as in the previous case. Introduce the adjoint body of the same shape and elastic stiffness or compliance matrices, but satisfying the boundary conditions (9) and with imposed body force and initial stress fields specified by (15). Starting from (16), we obtain

$$\delta G_2 = (Z_{2T}^k)_S \delta a_k, \tag{30}$$

where

$$(Z_{2T}^k)_S = \int [\phi \delta_{kj} - \sigma_{ij,k} u_i^a + \sigma'_{ij} u_{i,k}] n_j \, dS, \quad (k = 1, 2, 3). \tag{31}$$

Thus  $Z_{2T}^k$  is expressed analogously as  $Z_{1T}^k$  with  $\psi(\sigma, \mathbf{u})$  replaced by  $\phi(\boldsymbol{\varepsilon}, \mathbf{u})$ . Theorem 1 applies for the integral (31); thus

$$(Z_{2T}^k)_S = 0 \tag{32}$$

for any closed surface within the homogeneous body.

### 3.2. Rotation of body domain

Consider now the case where the body is rotated in the vicinity of its equilibrium position, and denote the infinitesimal rotation vector by  $\delta \omega_p$ . The external tractions and surface displacements are also rotated correspondingly. The variation of point position is given by

$$x_k^* = x_k + \delta \varphi_k = x_k + e_{kpl} x_l \delta \omega_p = (\delta_{kl} + e_{kpl} \delta \omega_p) x_l = \alpha_{kl} x_l, \tag{33}$$

and

$$\delta \varphi_k = e_{kpl} x_l \delta \omega_p, \quad \alpha_{kl} = \delta_{kl} + e_{kpl} \delta \omega_p, \tag{34}$$

where  $e_{kpl}$  denotes the permutation tensor. The variation of the unit vector  $\mathbf{n}$  normal to

the boundary surface  $S$  is expressed as follows, cf. Ref. [16]:

$$\delta n_j = n_j n_k n_l \delta \varphi_{k,l} - n_k \delta \varphi_{k,j} = (e_{kp} n_j n_k n_l - e_{kpj} n_k) \delta \omega_p = -e_{kpj} n_k \delta \omega_p. \quad (35)$$

The variation of the displacement field is

$$\delta u_i = \delta \bar{u}_i + u_{i,k} \delta \varphi_k = \delta \bar{u}_i + e_{kpl} u_{i,k} x_l \delta \omega_p. \quad (36)$$

On the other hand, when the displacement field is rotated, we have

$$\delta u_i = e_{ipl} u_l \delta \omega_p. \quad (37)$$

From (36) and (37), it follows that

$$\delta \bar{u}_i^0 = (e_{ipl} u_l - e_{kpl} u_{i,k} x_l) \delta \omega_p \quad \text{on } S_u. \quad (38)$$

On the loaded boundary, we have

$$\delta T_i^0 = e_{ipl} T_l^0 \delta \omega_p = e_{ipl} \sigma_{ij} n_j \delta \omega_p, \quad (39)$$

and since

$$\delta T_i^0 = \delta(\sigma_{ij} n_j) = \delta \sigma_{ij} n_j + \sigma_{ij} \delta n_j = \delta \bar{\sigma}_{ij} n_j + \sigma_{ij,k} n_j \delta \varphi_k + \sigma_{ij} \delta n_j, \quad (40)$$

we obtain on  $S_T$

$$\begin{aligned} \delta \bar{T}_i^0 &= \delta \bar{\sigma}_{ij} n_j = \delta T_i^0 - \sigma_{ij,k} n_j \delta \varphi_k - \sigma_{ij} \delta n_j = e_{ipl} \sigma_{ij} n_j \delta \omega_p \\ &\quad - e_{kpl} \sigma_{ij,k} x_l n_j \delta \omega_p + e_{kpl} \sigma_{ij} n_k \delta \omega_p = [e_{ipl} \sigma_{ij} n_j \\ &\quad + e_{kpl} (\sigma_{il} n_k - x_l \sigma_{ij,k} n_j)] \delta \omega_p. \end{aligned} \quad (41)$$

Using (33)–(41), the expression (14) for  $\delta G_1$  can be presented in the form

$$\begin{aligned} \delta G_1 &= e_{kpl} \left\{ \iint \left[ \psi x_l n_k + \frac{\partial h}{\partial u_i} u_{i,k} x_l + \frac{\partial h}{\partial T_i} \sigma_{ij,k} x_l n_j - \frac{\partial h}{\partial T_i} \sigma_{il} n_k \right] dS \right. \\ &\quad + \iint \left[ \left( \frac{\partial h}{\partial u_k} - \sigma'_{kj} n_j \right) u_l - \left( \frac{\partial h}{\partial u_i} - \sigma'_{ij} n_j \right) u_{i,k} x_l \right] dS_u + \iint \left[ \left( \frac{\partial h}{\partial T_k} + u_k^0 \right) \sigma_{ij} n_j \right. \\ &\quad \left. + \left( \frac{\partial h}{\partial T_i} + u_i^0 \right) (\sigma_{il} n_k - x_l \sigma_{ij,k} n_j) \right] dS_T \left. \right\} \delta \omega_p \\ &= \left\{ e_{kpl} \int \left[ \psi x_l \delta_{jk} + \sigma_{lj} u_k^0 + \delta_{jk} \sigma_{il} u_i^0 - x_l \sigma_{ij,k} u_i^0 - \sigma'_{kj} u_l \right. \right. \\ &\quad \left. + x_l \sigma'_{ij} u_{i,k} n_j \right] dS + e_{kpl} \int \left[ \left( \frac{\partial h}{\partial u_i} - T_i^{00} \right) u_{i,k} x_l + T_k^{00} u_i + \frac{\partial h}{\partial T_k} T_l^0 \right] dS_T \\ &\quad + e_{kpl} \int \left[ \left( \frac{\partial h}{\partial T_i} + u_i^0 \right) \sigma_{ij,k} x_l n_j - \left( \frac{\partial h}{\partial T_i} + u_i^0 \right) \sigma_{il} n_k + \frac{\partial h}{\partial u_k} u_i^0 \right. \\ &\quad \left. - T_l u_k^0 \right] dS_u \left. \right\} \delta \omega_p. \end{aligned} \quad (42)$$

Assume now that  $h = h(u_k, T_k)$  is an isotropic function of its arguments, thus

$$h = h(M_1, M_2, M_3), \quad (43)$$



where

$$M_1 = T_k T_k, \quad M_2 = u_k u_k, \quad M_3 = T_k u_k \quad (44)$$

are invariants of  $\mathbf{u}$  and  $\mathbf{T}$ . In view of (43), we have

$$\frac{\partial h}{\partial T_k} = \frac{\partial h}{\partial M_1} 2T_k + \frac{\partial h}{\partial M_3} u_k, \quad \frac{\partial h}{\partial u_k} = \frac{\partial h}{\partial M_3} T_k + \frac{\partial h}{\partial M_2} 2u_k. \quad (45)$$

Substituting (45) into (42) and accounting for boundary conditions (9), it is seen that the last two integrals of (42) vanish and the variation  $\delta G_1$  can be presented in the form

$$\delta G_1 = (Z^p_{1R})_S \delta \omega_p, \quad (46)$$

where

$$(Z^p_{1R})_S = e_{kpl} \int [\psi x_l \delta_{jk} + \sigma_{ij} u_k^a + \delta_{jk} \sigma_{il} u_i^a - x_l \sigma_{ij,k} u_i^a - \sigma'_{kj} u_l + x_l \sigma'_{ij} u_{i,k}] n_j \, dS. \quad (47)$$

Using the equalities

$$e_{kpl} \int \sigma_{ij,k} u_i^a x_l n_j \, dS = e_{kpl} \int (\sigma_{ij} u_{i,j}^a x_l n_k - \sigma_{ij} u_{i,k}^a x_l n_j + \sigma_{il} u_i^a n_k) \, dS \quad (48)$$

and

$$e_{kpl} \sigma'_{kj} u_l n_j = -e_{kpl} \sigma'_{ij} u_k n_j \quad (49)$$

the expression (47) for  $Z^p_{1R}$  can be presented in an equivalent form containing only gradients of displacement fields  $u_i$  and  $u_i^a$ , namely

$$(Z^p_{1R})_S = e_{kpl} \int [-\sigma_{iq} \varepsilon_{iq}^a x_l \delta_{kj} + \sigma_{ij} u_{i,k}^a x_l + \sigma_{ij} u_k^a + \sigma'_{ij} u_{i,k} x_l + \psi x_l \delta_{kj}] n_j \, dS. \quad (50)$$

The expression (50) for  $Z^p_{1R}$  can further be transformed into the volume integral

$$\begin{aligned} (Z^p_{1R})_S &= e_{kpl} \int [(\sigma_{ij} u_k^a)_{,j} + (\sigma_{il} u_i^a)_{,k} - (x_l \sigma_{ij,k} u_i^a)_{,j} - (\sigma'_{kj} u_l)_{,j} \\ &\quad + (x_l \sigma'_{ij} u_{i,k})_{,j} + (\psi x_l)_{,k}] \, dV = e_{kpl} \int [\sigma_{ij} u_{k,j}^a + \sigma_{il,k} u_i^a + \sigma_{il} u_{i,k}^a \\ &\quad - \sigma_{il,k} u_i^a - x_l \sigma_{ij,k} u_i^a - \sigma'_{kj,j} u_l - \sigma'_{kj} u_{l,j} + \sigma'_{il} u_{i,k} \\ &\quad + x_l \sigma'_{ij,j} u_{i,k} + x_l \sigma'_{ij} u_{i,k,j} + \psi \delta_{ik} + \varepsilon_{ij}^a \sigma_{ij,k} x_l + f_i^a u_{i,k} x_l] \, dV \\ &= e_{kpl} \int [2\sigma_{il} \varepsilon_{ik}^a + f_k^a u_l - \sigma'_{ki} u_{l,i} - \sigma'_{ik} u_{i,k}] \, dV \\ &= e_{kpl} \int [2\sigma_{il} \varepsilon_{ik}^a + f_k^a u_l - 2\sigma'_{ik} \varepsilon_{il}] \, dV. \end{aligned} \quad (51)$$

In writing (50), it is assumed that the body considered is *homogeneous*. Using now Hooke's law for an isotropic body,

$$\sigma'_{ik} = \lambda \varepsilon_{mm}^a \delta_{ik} + 2\mu \varepsilon'_{ik}, \quad (52)$$

where  $\lambda$  and  $\mu$  are Lamé constants, we obtain

$$e_{kpl}\sigma'_{ik}\varepsilon_{il} = e_{kpl}(\lambda\varepsilon'_{mm}\delta_{kl} + 2\mu\varepsilon'_{ik})\varepsilon_{il} = e_{kpl}2\mu\varepsilon'_{ik}\varepsilon_{il} \\ = e_{kpl}2\mu\varepsilon'_{ik}\left(-\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{mm}\delta_{il} + \frac{1}{\mu}\sigma_{il}\right) = e_{kpl}\varepsilon'_{ik}\sigma_{il}, \quad (53)$$

and the expression for  $Z^p_{1R}$  takes the form

$$(Z^p_{1R})_S = e_{kpl} \int (2\sigma_{il}\varepsilon^a_{ik} + f^a_k u_l - 2\sigma_{il}\varepsilon'_{ik}) \, dV = e_{kpl} \int (2\varepsilon^i_{ik}\sigma_{il} + f^a_k u_l) \, dV. \quad (54)$$

Assume now that  $\psi(\sigma, \mathbf{u}) = \psi_1(\sigma) + \psi_2(\mathbf{u})$  where  $\psi_1(\sigma)$  and  $\psi_2(\mathbf{u})$  are isotropic functions of stress and displacement; thus,

$$\psi_1 = \psi_1(\sigma_{ij}) = \psi_1(J_1, J_2, J_3), \quad \psi_2 = \psi_2(u_i) = \psi_2(I_u), \quad (55)$$

where  $J_1, J_2, J_3$  are the stress tensor invariants and  $I_u$  is the displacement vector invariant, that is

$$J_1 = \sigma_{ii}, \quad J_2 = \frac{1}{2}(\sigma_{ij}\sigma_{ji} - \sigma_{ii}\sigma_{jj}), \\ J_3 = \frac{1}{6}e_{mnq}e_{rst}\sigma_{mr}\sigma_{ns}\sigma_{qt}, \\ I_u = u_i u_i. \quad (56)$$

With these assumptions, the following equalities now occur:

$$\varepsilon^i_{ik} = \frac{\partial\psi_1}{\partial\sigma_{ik}} = \frac{\partial\psi_1}{\partial J_m} \frac{\partial J_m}{\partial\sigma_{ik}}, \quad f^a_k = \frac{\partial\psi_2}{\partial u_k} = \frac{\partial\psi_2}{\partial I_u} \frac{\partial I_u}{\partial u_k} = 2 \frac{\partial\psi_2}{\partial I_u} u_k, \quad (57)$$

where

$$\frac{\partial J_1}{\partial\sigma_{ik}} = \delta_{ik}, \quad \frac{\partial J_2}{\partial\sigma_{ik}} = \sigma_{ik} - \delta_{ik}\sigma_{qq}, \quad \frac{\partial J_3}{\partial\sigma_{ik}} = \frac{1}{2}e_{iqr}e_{kst}\sigma_{qs}\sigma_{rt}. \quad (58)$$

The expression for  $Z^p_{1R}$  can be presented in the form

$$(Z^p_{1R})_S = 2e_{kpl} \int \left\{ \left[ \frac{\partial\psi}{\partial J_1} \delta_{ik} + \frac{\partial\psi}{\partial J_2} (\sigma_{ik} - \delta_{ik}\sigma_{qq}) + \frac{\partial\psi}{\partial J_3} \frac{1}{2} e_{iqr} e_{kst} \sigma_{qs} \sigma_{rt} \right] \sigma_{il} \right. \\ \left. + \frac{\partial\psi}{\partial I_u} u_k u_l \right\} \, dV = 2 \int \left\{ e_{kpl} \left[ \frac{\partial\psi}{\partial J_1} \sigma_{kl} + \frac{\partial\psi}{\partial J_2} (\sigma_{ik}\sigma_{il} - \sigma_{kl}\sigma_{qq}) + \frac{\partial\psi}{\partial I_u} u_k u_l \right] \right. \\ \left. + \frac{\partial\psi}{\partial J_3} e_{iqr} \sigma_{pq} \sigma_{rl} \sigma_{il} \right\} \, dV = 0. \quad (59)$$

Thus, the following theorem can now be stated.

**THEOREM 2:** For a linear elastic and isotropic body, the surface integral

$$(Z^p_{1R})_S = e_{kpl} \int [\sigma_{ij}u^a_k + \delta_{jk}\sigma_{il}u^a_i - x_l\sigma_{ij,k}u^a_i - \sigma^a_{kj}u_l + x_l\sigma^a_{ij}u_{i,k} + \delta_{jk}\psi x_l] n_j \, dS \quad (60)$$

vanishes for any closed surface within the body, provided the adjoint system satisfies (9) and (10) and the functions  $\psi = \psi_1(\sigma) + \psi_2(\mathbf{u})$  and  $h(\mathbf{u}, \mathbf{T})$  are isotropic functions of their arguments.

The variation of the functional  $G_2$  is expressed similarly to (46) and (47) with  $\psi(\boldsymbol{\sigma}, \mathbf{u})$  replaced by  $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = \phi_1(\boldsymbol{\varepsilon}) + \phi_2(\mathbf{u})$ , thus

$$\delta G_2 = (\mathbf{Z}_{2R}^p)_S \delta \omega_p, \quad (61)$$

where  $\mathbf{Z}_{2R}^p$  is expressed identically as (50) with  $\psi_1(\boldsymbol{\sigma})$  replaced by  $\phi_1(\boldsymbol{\varepsilon})$ .

### 3.3. Expansion or contraction of body domain

In this section, we shall confine ourselves to particular cases of general functionals (1) and (2). Consider first the transformation

$$x_k^* = (1 + \delta\eta)x_k = (1 + \delta\eta)\delta_{ki}x_i, \quad \delta\varphi_k = x_k\delta\eta. \quad (62)$$

The matrix of coordinate transformation is now

$$\alpha_{ki} = (1 + \delta\eta)\delta_{ki}. \quad (63)$$

The variation of the exterior normal to the boundary surface now vanishes, thus

$$\delta n_j = n_j n_k n_l \delta\varphi_{k,l} - n_k \delta\varphi_{k,j} = (n_j n_k n_l \delta_{kl} - n_k \delta_{kj})\delta\eta = n_j(n_k n_k - 1)\delta\eta = 0, \quad (64)$$

and the variation of the displacement field is

$$u_i^* = (1 + \xi\delta\eta)u_i, \quad \delta u_i = \xi u_i \delta\eta = \delta \bar{u}_i + u_{i,k} x_k \delta\eta, \quad (65)$$

so that

$$\delta \bar{u}_i = (\xi u_i - x_k u_{i,k})\delta\eta, \quad (66)$$

where  $\xi$  is a constant to be determined. The variation of strain field follows from (66), namely

$$\begin{aligned} \frac{\partial u_i^*}{\partial x_j^*} &= \frac{\partial[(1 + \xi\delta\eta)u_i]}{\partial[(1 + \delta\eta)x_j]} = \frac{1 + \xi\delta\eta}{1 + \delta\eta} \frac{\partial u_i}{\partial x_j} \cong [1 + (\xi - 1)\delta\eta] \frac{\partial u_i}{\partial x_j}, \\ \varepsilon_{ij}^* &= [1 + (\xi - 1)\delta\eta]\varepsilon_{ij}, \quad \delta\varepsilon_{ij} = (\xi - 1)\varepsilon_{ij}\delta\eta = \delta \bar{\varepsilon}_{ij} + x_k \varepsilon_{ij,k} \delta\eta, \end{aligned} \quad (67)$$

and

$$\delta \bar{\varepsilon}_{ij} = [(\xi - 1)\varepsilon_{ij} - x_k \varepsilon_{ij,k}]\delta\eta. \quad (68)$$

The stress variation is expressed similarly, thus

$$\begin{aligned} \sigma_{ij}^* &= [1 + (\xi - 1)\delta\eta]\sigma_{ij}, \quad \delta \bar{\sigma}_{ij} = [(\xi - 1)\sigma_{ij} - x_k \sigma_{ij,k}]\delta\eta, \\ \delta \bar{\sigma}_{ij} n_j &= [(\xi - 1)\sigma_{ij} - x_k \sigma_{ij,k}]n_j \delta\eta. \end{aligned} \quad (69)$$

Consider now the particular form of the functional (1) for which  $\psi(\boldsymbol{\sigma}, \mathbf{u}) = \psi_1(\boldsymbol{\sigma})$ ,  $h(\mathbf{u}, \mathbf{T}) = 0$ , thus

$$G_\sigma = \int \psi_1(\boldsymbol{\sigma}) dV. \quad (70)$$

Assume  $\psi(\boldsymbol{\sigma})$  to be a homogeneous function of stress of order  $p$ ; thus,

$$\psi_1(t\boldsymbol{\sigma}) = t^p \psi_1(\boldsymbol{\sigma}), \quad (71)$$

and

$$\psi_1(\sigma^*) = [1 + (\xi - 1)\delta\eta]^p \psi_1(\sigma), \quad (72)$$

$$\delta\psi_1(\sigma) = \{[1 + (\xi - 1)\delta\eta]^p - 1\} \psi_1(\sigma).$$

The constant  $\xi$  is now determined by requiring the invariance of  $\psi \, dV$  under the transformation (62), that is

$$\delta(\psi_1 \, dV) = \delta\psi_1 \, dV + \psi_1 \delta(dV) = 0, \quad (73)$$

where  $\delta(dV) = 3\delta\eta \, dV$  for a three-dimensional case and  $\delta(dV) = 2\delta\eta \, dV$  for the plane case. In view of (71), the invariance condition for the three-dimensional scale change is expressed as follows:

$$\{[1 + (\xi - 1)\delta\eta]^p - 1\} \psi_1 \, dV + 3\delta\eta \psi_1 \, dV = 0. \quad (74)$$

Considering only linear terms of  $\delta\eta$ , from (74) we obtain

$$1 + p(\xi - 1)\delta\eta - 1 + 3\delta\eta = 0, \quad \xi = \frac{p-3}{p}. \quad (75)$$

Similarly, for the plane case, we have

$$[1 + p(\xi - 1)\delta\eta]^p - 1 + 2\delta\eta = 0, \quad \xi = \frac{p-2}{p}. \quad (76)$$

The variation of  $G_\sigma$  is now expressed as follows in the case of three-dimensional problem:

$$\delta G_\sigma = (Z_{\sigma E})_S \delta\eta, \quad (77)$$

where

$$(Z_{\sigma E})_S = \iint \left[ -\frac{3}{p} \sigma_{ij} u_i^a - x_k \sigma_{ij,k} u_i^a - \frac{p-3}{p} \sigma'_{ij} u_i + x_k \sigma'_{ij} u_{i,k} + \delta_{jk} x_k \psi_1 \right] n_j \, dS. \quad (78)$$

The equivalent form of  $Z_{\sigma E}$  containing only gradients of displacement fields  $u_i$  and  $u_i^a$  can be expressed as follows:

$$(Z_{\sigma E})_S = \iint \left[ \frac{2p-3}{p} \sigma_{ij} u_i^a - x_k \sigma_{ij} \varepsilon_{ij}^a \delta_{kj} + x_k \sigma_{ij} u_{i,k}^a - \frac{p-3}{p} \sigma'_{ij} u_i + x_k \sigma'_{ij} u_{i,k} + \delta_{jk} x_k \psi_1 \right] n_j \, dS. \quad (79)$$

Now, let us demonstrate that the integral (78) vanishes for any closed surface  $S$  within a homogeneous body. Transforming (78) into a volume integral, we obtain

$$\begin{aligned} (Z_{\sigma E})_S &= \iint \left[ -\frac{3}{p} (\sigma_{ij} u_i^a)_{,j} - (x_k \sigma_{ij,k} u_i^a)_{,j} - \frac{p-3}{p} (\sigma'_{ij} u_i)_{,j} + (x_k \sigma'_{ij} u_{i,k})_{,j} + (x_k \psi_1)_{,k} \right] dV \\ &= \iint \left[ -\frac{3}{p} \sigma_{ij} \varepsilon_{ij}^a - x_k \sigma_{ij,k} \varepsilon_{ij}^a - \frac{p-3}{p} \sigma'_{ij} \varepsilon_{ij} + \sigma'_{ij} \varepsilon_{ij} + x_k \sigma'_{ij} \varepsilon_{ij,k} + x_k \frac{\partial \psi_1}{\partial \sigma_{ij}} \sigma_{ij,k} + 3\psi_1 \right] dV \\ &= \iint \left[ -\frac{3}{p} \sigma_{ij} \varepsilon_{ij}^a - x_k (\sigma_{ij,k} \varepsilon_{ij}^a - \varepsilon'_{ij} \sigma_{ij,k} - \varepsilon_{ij}^a \sigma_{ij,k}) + \frac{3}{p} \varepsilon'_{ij} \sigma_{ij} + 3\psi_1 \right] dV \\ &= 3 \iint \left( -\frac{1}{p} \varepsilon'_{ij} \sigma_{ij} + \psi_1 \right) dV = \frac{3}{p} \iint \left( -\frac{\partial \psi_1}{\partial \sigma_{ij}} \sigma_{ij} + p\psi_1 \right) dV = 0, \end{aligned} \quad (80)$$

since for a homogeneous function of order  $p$  we have

$$\frac{\partial \psi_1}{\partial \sigma_{ij}} \sigma_{ij} = p \psi_1(\sigma_{ij}). \tag{81}$$

Similarly, for the functional (2), we have

$$G_\epsilon = \int \phi_1(\epsilon) \, dV, \quad \delta G_\epsilon = (Z_{\epsilon E})_S \delta \eta, \tag{82}$$

where

$$(Z_{\epsilon E})_S = \int \left[ -\frac{3}{p} \sigma_{ij} u_i^q - x_k \sigma_{ij,k} u_i^q - \frac{p-3}{p} \sigma'_{ij} u_i + x_k \sigma'_{ij} u_{i,k} + \delta_{jk} x_k \phi_1 \right] n_j \, dS, \tag{83}$$

or  $Z_{\epsilon E}$  is expressed by (79) with  $\psi(\sigma)$  replaced by  $\phi(\epsilon)$  and  $(Z_{\epsilon E})_S = 0$ .

Consider now the displacement functional

$$G = \int \psi_2(\mathbf{u}) \, dV \tag{84}$$

specified over the body domain, where  $\psi_2(\mathbf{u})$  is a homogeneous function of order  $p$ . Similarly to the previous analysis, we have

$$\psi_2(\mathbf{u}^*) = (1 + \xi \delta \eta)^p \psi_2(\mathbf{u}), \quad \delta \psi_2(\mathbf{u}) = [(1 + \xi \delta \eta)^p - 1] \psi_2(\mathbf{u}). \tag{85}$$

Assuming invariance of  $(\psi_2 \, dV)$  under the transformation (62), we obtain the values of  $\xi$  for the three-dimensional and plane cases, namely

$$(\xi)_{3d} = -\frac{3}{p}, \quad (\xi)_{\text{plane}} = -\frac{2}{p}. \tag{86}$$

The local variations of boundary displacements on  $S_u$  and of surface tractions on  $S_T$ , in view of (66) and (68), are now expressed as follows for the three-dimensional transformation:

$$\delta \bar{u}_i^0 = \left( -\frac{3}{p} u_i + x_k u_{i,k} \right) \delta \eta \quad \text{on } S_u, \tag{87}$$

$$\delta \bar{T}_i^0 = -\left( \frac{3+p}{p} \sigma_{ij} + x_k \sigma_{ij,k} \right) n_j \delta \eta \quad \text{on } S_T,$$

and the variation of  $G$  can be briefly expressed using the general formula (14), as

$$\delta G = (Z_{\nu E})_S \delta \eta, \tag{88}$$

where

$$(Z_{\nu E})_S = \int \left[ \delta_{jk} x_k \psi_2 + \frac{3}{p} \sigma'_{ij} u_i + x_k \sigma'_{ij} u_{i,k} - \frac{3+p}{p} \sigma_{ij} u_i^q - x_k \sigma_{ij,k} u_i^q \right] n_j \, dS, \tag{89}$$

since now  $\sigma'_{ij} n_j = T_i^q = 0$  on  $S_T$  and  $u_i^q = 0$  on  $S_u$ . The invariance of (89) for a homogeneous

linear elastic body is shown by transforming (89) into a volume integral as follows:

$$\begin{aligned}
 (Z_{rE})_S &= \int \left[ (x_k \psi_2)_{,k} + \frac{3}{p} (\sigma'_{ij} u_i)_{,j} + (x_k \sigma'_{ij} u_{i,k})_{,j} - \frac{3+p}{p} (\sigma_{ij} u_i^a)_{,j} - (x_k \sigma_{ij,k} u_i^a)_{,j} \right] dV \\
 &= \int \left[ 3\psi_2 + x_k f_1^a u_{i,k} - \frac{3}{p} f_1^a u_i + \frac{3}{p} \sigma'_{ij} \varepsilon_{ij} + \sigma'_{ij} \varepsilon_{ij} - x_k f_1^a u_{i,k} + x_k \sigma'_{ij} \varepsilon_{ij,k} \right. \\
 &\quad \left. - \frac{3+p}{p} \sigma_{ij} \varepsilon_{ij}^a - x_k \sigma_{ij,k} \varepsilon_{ij}^a \right] dV = \int \left( 3\psi - \frac{3}{p} f_1^a u_i \right) dV = \frac{3}{p} \int \left( p\psi_2 - \frac{\partial \psi_2}{\partial u_i} u_i \right) dV = 0.
 \end{aligned}
 \tag{90}$$

Similar expressions will be obtained by considering surface displacement and traction functionals. Consider first the functional

$$G = \int h_1(\mathbf{T}) dS,
 \tag{91}$$

where  $h_1(\mathbf{T})$  is assumed to be a homogeneous function of order  $p$ . We have

$$h_1(\mathbf{T}^*) = [1 + (\xi - 1)\delta\eta]^p h_1(\mathbf{T}), \quad \delta h_1(\mathbf{T}) = \{[1 + (\xi - 1)\delta\eta]^p - 1\} h_1(\mathbf{T}).
 \tag{92}$$

Assume now the invariance of  $(h_1 dS)$  under the transformation (62) and note that

$$\delta(h_1 dS) = \delta h_1 dS + h_1 \delta(dS) = 0,
 \tag{93}$$

$$\delta(dS) = (\delta_{kl} \delta_{kl} - n_k n_l \delta_{kl}) \delta\eta = 2\delta\eta,$$

and  $\delta(dS) = \delta\eta$  in the plane case. The first condition (93) implies that

$$(\xi)_{3d} = \frac{p-2}{p}, \quad (\xi)_{\text{plane}} = \frac{p-1}{p},
 \tag{94}$$

and the variation of (91) takes the form

$$\delta G = (Z_{TE})_S \delta\eta,
 \tag{95}$$

where

$$(Z_{TE})_S = \int \left( \frac{p-2}{p} \sigma'_{ij} u_i + x_k \sigma'_{ij} u_{i,k} - \frac{2}{p} \sigma_{ij} u_i^a - x_k \sigma_{ij,k} u_i^a \right) n_j dS
 \tag{96}$$

for the three-dimensional case.

Similarly, for the functional

$$G = \int h_2(\mathbf{u}) dS,
 \tag{97}$$

where  $h_2(\mathbf{u})$  is a homogeneous function of order  $p$ , from (93) we obtain

$$(\xi)_{3d} = -\frac{2}{p}, \quad (\xi)_{\text{plane}} = -\frac{1}{p},
 \tag{98}$$

and

$$\delta G = (Z_{uE})_S \delta \eta, \tag{99}$$

where

$$(Z_{uE})_S = \int \left( \frac{2}{p} \sigma'_{ij} u_i + x_k \sigma'_{ij} u_{i,k} - \frac{p+2}{p} \sigma_{ij} u_i^a - x_k \sigma_{ij,k} u_i^a \right) n_j \, dS. \tag{100}$$

It is easy to show that (96) and (100) vanish for a linear elastic and homogeneous body. In fact, transforming (100) into a volume integral, we obtain

$$\begin{aligned} (Z_{uE})_S &= \int \left( \frac{2}{p} \sigma'_{ij} \varepsilon_{ij} + \sigma'_{ij} \varepsilon_{ij} + x_k \sigma'_{ij} \varepsilon_{ij,k} - \frac{p+2}{p} \sigma_{ij} \varepsilon_{ij}^a - x_k \sigma_{ij,k} \varepsilon_{ij}^a \right) dV \\ &= \int \left[ \frac{p+2}{p} (\sigma'_{ij} \varepsilon_{ij} - \sigma_{ij} \varepsilon_{ij}^a) + x_k (\sigma'_{ij} \varepsilon_{ij,k} - \sigma_{ij,k} \varepsilon_{ij}^a) \right] dV = 0, \end{aligned} \tag{101}$$

and the identical expression is obtained for the surface integral (96) when  $(p+2)/p$  is replaced by  $2/p$ .

**THEOREM 3:** For a linear elastic and homogeneous body, the surface integrals (79), (83), (89), (96) and (100) vanish for any closed surface within the body. The variation of the respective functionals (70), (82), (84), (91) and (97) associated with the scale change of the body therefore vanishes.

#### 4. PATH-INDEPENDENT INTEGRALS ASSOCIATED WITH POTENTIAL AND COMPLEMENTARY ENERGY VARIATIONS

The transition to the case where (1) or (2) coincide with the complementary or potential energy of the body can be obtained by specifying the adjoint systems and using general expressions for  $Z_T$ ,  $Z_R$  and  $Z_E$ . Consider first the particular case when only specified surface tractions  $\mathbf{T}^0$  or only surface displacements  $\mathbf{u}^0$  expend the work on the body. In the first case,  $\mathbf{u}^0 = 0$  on  $S_u$ ,  $\mathbf{T} = \mathbf{T}^0 \neq 0$  on  $S_T$ , and the potential and complementary energies are expressed as follows:

$$\Pi_u = \int U(\boldsymbol{\varepsilon}) \, dV - \int \mathbf{T}^0 \cdot \mathbf{u} \, dS_T = - \int U(\boldsymbol{\varepsilon}) \, dV, \tag{102}$$

and

$$\Pi_\sigma = \int W(\boldsymbol{\sigma}) \, dV - \int \mathbf{T} \cdot \mathbf{u}^0 \, dS_u = \int W(\boldsymbol{\sigma}) \, dV, \tag{103}$$

where  $U(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$ ,  $W(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$  are the specific stress and strain energies per unit volume. In the second case,  $\mathbf{T}^0 = 0$  on  $S_T$  and  $\mathbf{u} = \mathbf{u}^0 \neq 0$  on  $S_u$ , so that

$$\Pi_u = \int U(\boldsymbol{\varepsilon}) \, dV, \quad \Pi_\sigma = - \int W(\boldsymbol{\sigma}) \, dV. \tag{104}$$

Setting  $h(\mathbf{T}, \mathbf{u}) = 0$ ,  $\psi(\boldsymbol{\sigma}, \mathbf{u}) = W(\boldsymbol{\sigma})$ , and  $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = -U(\boldsymbol{\varepsilon})$ , the functionals (1) and (2) will be equivalent to (103) and (102). Similarly, setting  $\psi(\boldsymbol{\sigma}, \mathbf{u}) = -W(\boldsymbol{\sigma})$  and  $\phi(\boldsymbol{\varepsilon}, \mathbf{u}) = U(\boldsymbol{\varepsilon})$  in (1) and (2), we obtain the functionals (104).

The adjoint system associated with the potential energy  $\Pi_u$  for the case  $\mathbf{u}^0 = 0$  on  $S_u$  is now specified as follows:

$$\begin{aligned} \boldsymbol{\sigma}^i &= \frac{\partial \phi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial(-U)}{\partial \boldsymbol{\varepsilon}} = -\boldsymbol{\sigma} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u, \\ \boldsymbol{\sigma}^r &= 0, \quad \boldsymbol{\sigma}^a = \mathbf{D} \cdot \boldsymbol{\varepsilon}^a = -\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}^a = -\boldsymbol{\varepsilon}, \quad \mathbf{u}^a = -\mathbf{u}, \end{aligned} \quad (105)$$

whereas for the case  $\mathbf{T}^0 = 0$  on  $S_T$  we have

$$\begin{aligned} \boldsymbol{\sigma}^i &= \frac{\partial \phi}{\partial \boldsymbol{\sigma}} = \frac{\partial U}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u, \\ \boldsymbol{\sigma}^r &= -\boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^a = 0, \quad \mathbf{u}^a = \boldsymbol{\varepsilon}^a = 0. \end{aligned} \quad (106)$$

In view of (105) and (106), the general expressions for  $Z_{2T}^k$ ,  $Z_{2R}^l$  and  $Z_{\varepsilon E}$  given by (31), (50) and (83) take the following forms:

$$\begin{aligned} (Z_{2T}^k)_S &= \int (U \delta_{kj} - \sigma_{ij} u_{i,k}) n_j \, dS = 0, \\ (Z_{2R}^l)_S &= e_{kpl} \int (U x_l \delta_{kj} - \sigma_{ij} u_k - \sigma_{ij} u_{i,k} x_l) n_j \, dS = 0, \\ (Z_{\varepsilon E})_S &= \int (U x_k \delta_{jk} - \sigma_{ij} u_{i,k} x_k - \frac{1}{2} \sigma_{ij} u_i) n_j \, dS = 0. \end{aligned} \quad (107)$$

The relations (107) constitute the set of conservation laws associated with invariance of the potential energy under translation, rotation and scale change of the domain of a homogeneous and isotropic body (isotropy is only required in the case of rotation). Consider now the complementary energy  $\Pi_\sigma$ . The adjoint system for the case  $\mathbf{u}^0 = 0$  on  $S_u$  is now specified as follows:

$$\begin{aligned} \boldsymbol{\varepsilon}^i &= \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \frac{\partial W}{\partial \boldsymbol{\sigma}} = \boldsymbol{\varepsilon} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u, \\ \boldsymbol{\sigma}^r &= 0, \quad \boldsymbol{\sigma}^a = \mathbf{D} \cdot \boldsymbol{\varepsilon}^a = \boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}^a = \boldsymbol{\varepsilon}, \quad \mathbf{u}^a = \mathbf{u}, \end{aligned} \quad (108)$$

whereas for the case  $\mathbf{T}^0 = 0$  on  $S_T$  we can write

$$\begin{aligned} \boldsymbol{\varepsilon}^i &= \frac{\partial \psi}{\partial \boldsymbol{\sigma}} = \frac{\partial(-W)}{\partial \boldsymbol{\sigma}} = -\boldsymbol{\varepsilon} \text{ within } V, \quad \mathbf{T}^{a_0} = 0 \text{ on } S_T, \quad \mathbf{u}^{a_0} = 0 \text{ on } S_u, \\ \boldsymbol{\sigma}^r &= \boldsymbol{\sigma}, \quad \boldsymbol{\sigma}^a = 0, \quad \boldsymbol{\varepsilon}^a = \mathbf{u}^a = 0. \end{aligned} \quad (109)$$

Using now (108) and (109) in the general expressions for  $Z_{1T}^k$ ,  $Z_{1R}^l$  and  $Z_{\sigma E}$  derived in the previous section, we obtain the conservation laws associated with the invariance of the complementary energy under translation, rotation and expansion of the domain of a homogeneous and isotropic body, namely

$$\begin{aligned} (Z_{1T}^k)_S &= \int (-W \delta_{kj} + \sigma_{ij} u_{i,k}) n_j \, dS = 0, \\ (Z_{1R}^l)_S &= e_{kpl} \int (-W x_l \delta_{kj} + \sigma_{ij} u_k + \sigma_{ij} u_{i,k} x_l) n_j \, dS = 0, \\ (Z_{\sigma E})_S &= \int (-W x_k \delta_{jk} + \sigma_{ij} u_{i,k} x_k + \frac{1}{2} \sigma_{ij} u_i) n_j \, dS = 0. \end{aligned} \quad (110)$$



Let us note that the conservation laws (107) and (110) are respectively equivalent to those derived earlier by Günther[3], Eshelby[1, 2], Knowles and Sternberg[4] and Bui[14].

It can be shown that the same path-independent functionals and conservation rules are obtained in the case of translation and rotation of the body when mixed boundary conditions with non-vanishing  $\mathbf{u}^0$  and  $\mathbf{T}^0$  on  $S_u$  and  $S_T$  are assumed.

5. CONSERVATION RULES FOR BILINEAR FUNCTIONALS

Bilinear functionals are useful in assessing the variation of local displacement or stress components due to shape variation. It was shown in Ref. [20] that they can be generated as singular cases of (1) and (2) assuming localized force or dislocation action on the adjoint body. However, it is convenient to consider this class of functionals separately. Consider a linear elastic body loaded by surface traction fields  $\mathbf{T}_1^0$  and  $\mathbf{T}_2^0$  on  $S_T$ , and with specified displacements  $\mathbf{u}_1^0$  and  $\mathbf{u}_2^0$  on  $S_u$ , respectively. Denote the corresponding state fields by  $\sigma_1, \epsilon_1, \mathbf{u}_1$  and  $\sigma_2, \epsilon_2, \mathbf{u}_2$  satisfying conditions of equilibrium, compatibility and constitutive relations. Consider now the functionals, cf. Shield and Prager[21],

$$\begin{aligned}
 B_1 &= \int \frac{1}{2} \sigma_1 \cdot \epsilon_2 \, dV + \int \frac{1}{2} \sigma_2 \cdot \epsilon_1 \, dV - \int \mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T - \int \mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T \\
 &= \int \tilde{U}(\epsilon_1, \epsilon_2) \, dV - \int \mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T - \int \mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T,
 \end{aligned}
 \tag{111}$$

and

$$\begin{aligned}
 B_2 &= \int \frac{1}{2} \sigma_1 \cdot \epsilon_2 \, dV + \int \frac{1}{2} \sigma_2 \cdot \epsilon_1 \, dV - \int \mathbf{T}_1 \cdot \mathbf{u}_2^0 \, dS_u - \int \mathbf{T}_2 \cdot \mathbf{u}_1^0 \, dS_u \\
 &= \int \tilde{W}(\sigma_1, \sigma_2) \, dV - \int \mathbf{T}_1 \cdot \mathbf{u}_2^0 \, dS_u - \int \mathbf{T}_2 \cdot \mathbf{u}_1^0 \, dS_u,
 \end{aligned}
 \tag{112}$$

where  $\tilde{U}(\epsilon_1, \epsilon_2) = \epsilon_1 \cdot \mathbf{D} \cdot \epsilon_2 = \epsilon_2 \cdot \mathbf{D} \cdot \epsilon_1$  and  $\tilde{W}(\sigma_1, \sigma_2) = \sigma_1 \cdot \mathbf{C} \cdot \sigma_2 = \sigma_2 \cdot \mathbf{C} \cdot \sigma_1$  are the specific mutual strain and stress energies, so that

$$\sigma_1 = \frac{\partial \tilde{U}}{\partial \epsilon_2}, \quad \sigma_2 = \frac{\partial \tilde{U}}{\partial \epsilon_1}, \quad \epsilon_1 = \frac{\partial \tilde{W}}{\partial \sigma_2}, \quad \epsilon_2 = \frac{\partial \tilde{W}}{\partial \sigma_1}.
 \tag{113}$$

The variation of  $B_1$  due to shape variation is expressed similarly to (6), namely

$$\begin{aligned}
 \delta B_1 &= \int \left( \frac{\partial \tilde{U}}{\partial \epsilon_1} \cdot \delta \epsilon_1 + \frac{\partial \tilde{U}}{\partial \epsilon_2} \cdot \delta \epsilon_2 \right) \, dV + \int \tilde{U} n_k \delta \varphi_k \, dS - \int \delta \mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T \\
 &\quad - \int \mathbf{T}_1^0 \cdot (\delta \bar{\mathbf{u}}_2 + \mathbf{u}_{2,k} \delta \varphi_k) \, dS_T - \int \delta \mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T \\
 &\quad - \int \mathbf{T}_2^0 \cdot (\delta \bar{\mathbf{u}}_1 + \mathbf{u}_{1,k} \delta \varphi_k) \, dS_T \\
 &\quad - \int (\mathbf{T}_1^0 \cdot \mathbf{u}_2 + \mathbf{T}_2^0 \cdot \mathbf{u}_1) (\delta_{kl} - n_k n_l) \delta \varphi_{k,l} \, dS_T.
 \end{aligned}
 \tag{114}$$

Using the virtual work equation, we obtain

$$\begin{aligned} \delta B_1 = & \int (\tilde{U}n_k - \mathbf{T}_1 \cdot \mathbf{u}_{2,k} - \mathbf{T}_2 \cdot \mathbf{u}_{1,k}) \delta \varphi_k \, dS \\ & - \int (\mathbf{T}_1^0 \cdot \mathbf{u}_2 + \mathbf{T}_2^0 \cdot \mathbf{u}_1) (\delta_{kl} - n_k n_l) \delta \varphi_{k,l} \, dS_T \\ & - \int \delta \mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T - \int \delta \mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T + \int \mathbf{T}_1 \cdot \delta \mathbf{u}_2^0 \, dS_u + \int \mathbf{T}_2 \cdot \delta \mathbf{u}_1^0 \, dS_u, \end{aligned} \tag{115}$$

and a similar expression is obtained for the functional  $B_2$ :

$$\begin{aligned} \delta B_2 = & \int (-\tilde{W}n_k + \mathbf{T}_1 \cdot \mathbf{u}_{2,k} + \mathbf{T}_2 \cdot \mathbf{u}_{1,k}) \delta \varphi_k \, dS \\ & + \int (\mathbf{T}_1^0 \cdot \mathbf{u}_2 + \mathbf{T}_2^0 \cdot \mathbf{u}_1) (\delta_{kl} - n_k n_l) \delta \varphi_{k,l} \, dS_T + \int \delta \mathbf{T}_1^0 \cdot \mathbf{u}_2 \, dS_T \\ & + \int \delta \mathbf{T}_2^0 \cdot \mathbf{u}_1 \, dS_T - \int \mathbf{T}_1 \cdot \delta \mathbf{u}_2^0 \, dS_u - \int \mathbf{T}_2 \cdot \delta \mathbf{u}_1^0 \, dS_u. \end{aligned} \tag{116}$$

Consider now the translation of body domain:  $\delta \varphi_k = \delta a_k = \text{const.}$ ,  $\delta \mathbf{u}_1^0 = \delta \mathbf{u}_2^0 = 0$  on  $S_u$ ,  $\delta \mathbf{T}_1^0 = \delta \mathbf{T}_2^0 = 0$  on  $S_T$ . Similarly to (21), from (115) it follows that

$$\delta B_1 = (M_{1T}^k)_S \delta a_k, \tag{117}$$

where

$$(M_{1T}^k)_S = \int (\tilde{U}n_k - \mathbf{T}_1 \cdot \mathbf{u}_{2,k} - \mathbf{T}_2 \cdot \mathbf{u}_{1,k}) \, dS = \int (\tilde{U}\delta_{kj} - \sigma_{1ij}u_{2i,k} - \sigma_{2ij}u_{1i,k})n_j \, dS, \tag{118}$$

and  $(M_{1T}^k)_S = 0$  for any closed surface  $S$  within a homogeneous elastic body. It is seen that (118) is identical to (22) and also to the path-independent integral derived by Chen and Shield[18]. Similarly, the variation of  $B_2$  is expressed as follows:

$$\delta B_2 = (M_{2T}^k)_S \delta a_k = -(M_{1T}^k)_S \delta a_k, \tag{119}$$

where the path-independent integral is expressed as

$$(M_{2T}^k)_S = \int (-\tilde{W}\delta_{kj} + \sigma_{1ij}u_{2i,k} + \sigma_{2ij}u_{1i,k})n_j \, dS, \tag{120}$$

and it is equivalent to (31).

The case of rotation can be treated similarly as before, and the path-independent integrals are specified by (50). It is thus seen that the variations of the bilinear functionals are expressed in terms of the same path-independent integrals as the variations of  $G_1$  and  $G_2$ .

### 6. ON APPLICATION OF PATH-INDEPENDENT INTEGRALS

The derived path-independent integrals can be applied in sensitivity analysis with respect to translation, rotation or expansion of internal defects such as cracks, cavities or inclusions. In fact, the functionals (1) and (2) can be given different interpretations depend-

ing on the type of problem considered. For instance, they can represent local or averaged stress measure or distance norm between measured and theoretical values of strains and displacements within the body. Thus, these functionals can be used in identification of the positions and sizes of defects based on available measurement data. Enclosing the defect by a surface  $S_i$ , the variation of any functional  $G_1$  or  $G_2$  can be determined by calculating the respective path-independent integral along any surface  $S_i$  including the defect surface  $S_0$  or the external boundary surface  $S$ . Another application, already indicated in Refs. [18, 19], would be associated with separating and determining mixed-mode singularities at crack tips.

We will now illustrate the applicability of the derived conservation rules by considering two simple examples.

*Example 1.* Consider a thin circular disk of external radius  $b$  and with an internal hole of radius  $a$ , loaded by the pressure  $p$  at the external perimeter  $r = b$ . The stress and displacement states within the disk referred to the polar coordinate system  $r, \theta$  are as follows:

$$\begin{aligned} \sigma_r &= k\left(\frac{a^2}{r^2} - 1\right), & \sigma_\theta &= -k\left(\frac{a^2}{r^2} + 1\right), & k &= \frac{pb^2}{b^2 - a^2}, \\ u_r &= -\frac{k}{E_r} \left[ (1 + \nu) \frac{a^2}{r^2} + (1 - \nu) \right], & u_\theta &= 0, \end{aligned} \tag{121}$$

where  $E$  and  $\nu$  denote Young's modulus and Poisson's ratio, respectively. We consider now the functional

$$G = \int \psi(\sigma) \, dA = \int \sigma_\epsilon^2 \, dA = \int (\sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2) \, dA, \tag{122}$$

and determine its variation associated with an expansion of the internal hole. According to (77) and (78), this variation can be expressed by a path-independent integral over the external perimeter. In the two-dimensional case, we have  $p = 2$ ,  $\xi = 0$ , and

$$\delta G = (Z_{\sigma E})_S \delta \eta, \quad (Z_{\sigma E})_S = b \int (\psi - \sigma \cdot \epsilon^a + \mathbf{T} \cdot \mathbf{u}_{,n}^a - \mathbf{T}^a \cdot \mathbf{u}_{,n}) \, dS_T. \tag{123}$$

The adjoint body is subject to the initial strain field

$$\epsilon_r^i = \frac{\partial \psi}{\partial \sigma_r} = 2\sigma_r - \sigma_\theta = k\left(\frac{3a^2}{r^2} - 1\right), \quad \epsilon_\theta^i = \frac{\partial \psi}{\partial \sigma_\theta} = 2\sigma_\theta - \sigma_r = -k\left(\frac{3a^2}{r^2} + 1\right), \tag{124}$$

with the boundary conditions  $T_r^a(b) = \sigma_r^a(b) = 0$ ,  $T_r^a(a) = \sigma_r^a(a) = 0$ . The solution of the adjoint problem provides the stress, strain and displacement fields

$$\begin{aligned} u_r^a &= -kr\left(\frac{3a^2}{r^2} + 1\right), & u_\theta^a &= 0, & \sigma_r^a &= 0, & \sigma_\theta^a &= 0, \\ \epsilon_r^a &= k\left(\frac{3a^2}{r^2} - 1\right), & \epsilon_\theta^a &= -k\left(\frac{3a^2}{r^2} + 1\right), \end{aligned} \tag{125}$$

and

$$(Z_{\sigma E})_S = 2\pi b^2(\sigma_r^a - \sigma_\theta^a) = -8\pi k^2 a^2, \quad \delta G = -8\pi k^2 a^2 \delta \eta = 8\pi k^2 a \delta a, \tag{126}$$

since  $\delta a = -a \delta \eta$ , where  $\delta a > 0$  corresponds to the expansion of hole.

When the displacement functional

$$G = \int h(\mathbf{u}) \, dA = \frac{1}{2} \int u_r^2 \, dA \tag{127}$$

is considered, the adjoint structure is loaded by the body force field

$$f_r^a = \frac{\partial h}{\partial u_r} = u_r, \quad f_\theta^a = 0, \tag{128}$$

and, according to (88) and (89), the variation of (127) is expressed by a surface integral over the external perimeter; thus,

$$\delta G = (Z_{vE})_S \delta \eta = -\frac{1}{a} (Z_{vE})_S \delta a, \tag{129}$$

$$(Z_{vE})_S = \int (hb - \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^a b + \mathbf{T} \cdot \mathbf{u}_{,n}^a b - \mathbf{T} \cdot \mathbf{u}^a) \, dS_T = \pi b [u_r^2(b) + 4ku_r^a(b)],$$

where the displacement field for the adjoint body is obtained in the form

$$u_r^a = \frac{(1-\nu^2)k}{2E^2} \left[ \frac{1}{4} (1-\nu)r^3 + (1+\nu)a^2 r \left( \ln r - \frac{1}{2} \right) + C_1 r + \frac{2}{r} C_2 \right],$$

$$C_1 = - \left\{ \left[ \frac{1}{4} \frac{(3+\nu)(1-\nu)}{1+\nu} + \frac{1}{2}(1-\nu) \right] (a^2 + b^2) + (1+\nu) \frac{a^2(b^2 \ln b - a^2 \ln a)}{b^2 - a^2} \right\}, \tag{130}$$

$$C_2 = -a^2 b^2 \left[ \frac{1}{4} (3+\nu) + \frac{(1+\nu)^2}{1-\nu} \frac{a^2 (\ln b - \ln a)}{b^2 - a^2} \right].$$

*Example 2.* This example is related to generalization of the *J*-integral in fracture mechanics[12–14] associated with the potential energy release rate due to crack propagation. Consider a plane strain or stress case and a notch or crack with flat boundary portions oriented along the *x*-axis, cf. Fig. 3. For a curve  $\Gamma$  enclosing the notch tip and ending on flat notch surfaces, the integral (24) can be expressed as follows:

$$(Z_{1T})_\Gamma = \int (\psi - \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^a) \, dy + \int \left( \mathbf{T} \cdot \frac{\partial \mathbf{u}^a}{\partial x} + \mathbf{T}^a \cdot \frac{\partial \mathbf{u}}{\partial x} \right) \, ds, \tag{131}$$

where  $ds$  is the arc length element of  $\Gamma$  and  $dy$  denotes its projection on the *y*-axis of the *x, y*-Cartesian coordinate system. Similarly as for the *J*-integral[12], it is easy to prove path independence of this integral. In fact, considering any other curve  $\Gamma'$  ending at 3 and 4 on the flat notch portions, the integral over the whole closed path  $\Gamma_{1-2} + 2-4 + \Gamma'_{4-3} + 3-1$  vanishes in view of Theorem 1 since this path encloses the homogeneous material. Moreover, on flat portions 2–3 and 1–4 we have  $dy = 0$ ,  $\mathbf{T} = \mathbf{T}^a = 0$  and the expression in (131) vanishes on these portions. Thus,  $(Z_{1T})_{\Gamma'} = (Z_{1T})_\Gamma$ , and the integral (131) is path-independent. A similar property occurs for the integral  $(Z_{2T})_\Gamma$  specified by (31). Now, if the path coincides with the curvilinear portion  $\Gamma_0$  of the notch tip, the integral (131) takes the form

$$(Z_{1T})_\Gamma = \int (\psi - \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}^a) \, dy, \tag{132}$$

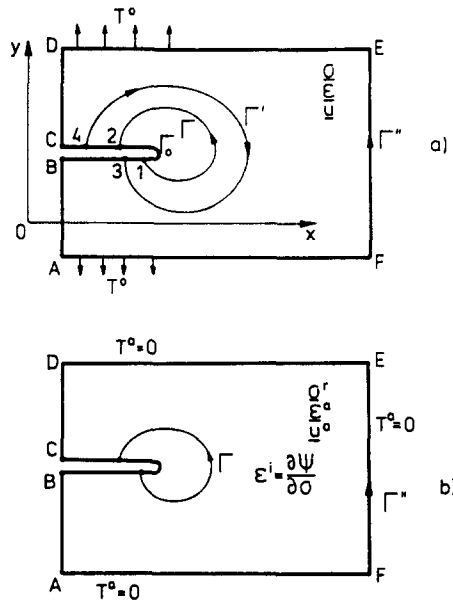


Fig. 3. Path-independent integral for a notched body in plane case: (a) body with a notch; (b) adjoint body subjected to initial strain field  $\epsilon^i$ .

and the variation of the functional (1) is given by

$$\delta G_i = (Z|_T)_{\Gamma_0} \delta a = (Z|_T)_{\Gamma} \delta a, \tag{133}$$

where  $\delta a$  denotes the infinitesimal translation of the notch tip along the  $x$ -axis. The familiar property of the  $J$ -integral associated with the potential energy variation is now preserved for any stress, strain or displacement functional.

Consider, for instance, the case where the external boundary of the body coincides with the  $x$ - and  $y$ -axes, cf. Fig. 3. Assume the boundaries  $AD$  and  $EF$  to be free and  $DE$ ,  $AF$  to be loaded by normal tractions. When the second term of (1) is neglected and  $\psi = \psi(\sigma)$ , the adjoint body is subjected to the initial strain field  $\epsilon^i$  specified by (10) with vanishing surface tractions  $T^a$  on the boundary. The integral (131) now takes the form

$$(Z|_T)_{ADEF} = \int_{EF} [\psi(\sigma_y) - \sigma_y \epsilon_y^i] dy - \int_{AD} [\psi(\sigma_y) - \sigma_y \epsilon_y^i] dy + \int_{DE} T_y \frac{\partial u_y^a}{\partial x} dx + \int_{AF} T_y \frac{\partial u_y^a}{\partial x} dx. \tag{134}$$

When the body is subjected to a uniformly distributed displacement on  $DE$  and  $AF$ , then  $\partial u_y^a / \partial x = 0$  and only two terms of (134) remain.

Whereas previous research effort in fracture mechanics was concentrated primarily on determining the variation of the potential energy associated with crack growth, the present method provides tools to determine variation of any functional of stress, strain or displacement. One class of functionals is associated with the problem of identification of damage or crack position and orientation. The two examples presented illustrate the idea that by calculating path-independent integrals along fixed contours, far from singularities, the variation of respective functionals can be assessed.

### 7. CONCLUDING REMARKS

In the present work, a new class of conservation laws and path-independent integrals was discussed. This class is associated with variation of arbitrary stress, strain and displacement functionals due to infinitesimal translation, rotation or scale change of an inhomogeneity

within the elastic body. The analysis was limited to linearly elastic material within small strain theory but the extension to non-linear case is possible by following the work on sensitivity analysis in non-linear elasticity [17]. Similarly as  $J$ -integral in fracture mechanics the derived path-independent integrals  $Z_{1T}^k$  or  $Z_{2T}^k$  can be applied in studying variation of any functional due to extension of plane cracks. The bilinear functionals discussed in Section 5 provide the same path-independent integrals with proper interpretation of the adjoint system. Their application was indicated in Refs. [18, 19] to problems of determining stress intensity factors at crack tips for mixed mode conditions.

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